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NOTE ON SOME THIRD-ORDER NONLINEAR SOLVERS FOR THE SOLUTION OF NONLINEAR EQUATIONS

Weonbae Kim*, Changbum Chun**, and Yong-Il Kim***

Abstract. In this note we extend one well-known iteration formula to derive new third-order methods for solving a nonlinear equation. The comparison with other methods is given.

1. Acceleration of convergence

This work is concerned with iteration functions to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$. Let $\phi(x)$ be an iteration function of order p for the solution α of the nonlinear equation $f(x) = 0$. That is, ϕ is a function which satisfies $\phi(\alpha) = \alpha,\ \phi^{(j)}(\alpha) = 0, j = 1, 2, ..., p-1,\ \phi^{(p)}(\alpha) \neq 0$ (see [1]). It is well known that if $\phi(x)$ is of order $p \geq 2$, then the iteration function defined by

$$\Phi(x) = \phi(x) - \frac{f[\phi(x)]}{f'(x)} \tag{1.1}$$

is of order at least $p + 1$ (see [1]). In this letter we extend this fact.

Theorem 1.1. *Assume that the function f is sufficiently smooth in a neighborhood of its root α, where $f'(\alpha) \neq 0$. Let $\phi(x)$ and $\psi(x)$ be iteration functions of order p and $q \geq p$, respectively. Then the iteration function*

$$\Phi(x) = \phi(x) - \frac{f[\psi(x)]}{f'(x)} \tag{1.2}$$

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Correspondence should be addressed to Changbum Chun, cbchun@skku.edu.
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is of order at least $p+1$.

Proof. The proof can be established in a similar fashion as that of Theorem 8-1 in [1]. For the sake of completeness, we give the proof in detail. Observe that $\Phi(\alpha) = \alpha$. It is easily verified that $\Phi'(\alpha) = 0$. By induction we prove that $\Phi^{(j)}(\alpha) = 0,\ j = 1, 2, ..., p$. Fix l as any integer such that $l \leq p$. Assume that $\Phi^{(j)}(\alpha) = 0,\ j = 1, 2, ..., l-1$. The definition of $\Phi(x)$ can be rewritten as

$$f'(x)\Phi(x) = f'(x)\phi(x) - f[\psi(x)]. \tag{1.3}$$

Differentiating (1.3) l times with respect to x yields

$$\begin{aligned} &\sum_{j=0}^{l} C[l,j] f^{(l-j+1)}(x)\Phi^{(j)}(x) \\ &= \sum_{j=0}^{l} C[l,j] f^{(l-j+1)}(x)\phi^{(j)}(x) - \frac{d^l f[\psi(x)]}{dx^l}, \end{aligned} \tag{1.4}$$

where $C[l,j]$ is a binomial coefficient. Set $x = \alpha$ in (1.4). By the induction assumption and the hypotheses that $\phi(x)$ is of order p, we have

$$\Phi^{(j)}(\alpha) = \phi^{(j)}(\alpha) = 0, \quad j = 1, 2, ..., l-1. \tag{1.5}$$

Since $\psi(x)$ is of order $q \geq p$, we have

$$\left.\frac{d^l f[\psi(x)]}{dx^l}\right|_{x=\alpha} = f'(\alpha)\psi^{(l)}(\alpha), \tag{1.6}$$

(See Theorem 2-14 in [1]). It then follows from (1.4) that

$$f'(\alpha)\Phi^{(l)}(\alpha) = f'(\alpha)\phi^{(l)}(\alpha) - f'(\alpha)\psi^{(l)}(\alpha). \tag{1.7}$$

This implies that $\Phi^{(l)}(\alpha) = 0$ since $f'(\alpha) \neq 0$. This completes the induction. Since l is any integer such that $l \leq p$, the proof is completed. □

Developing new third-order methods requiring at most three functional evaluations per step is still an important task in iteration calculus field. In this letter we are concerned with deriving new third-order methods by applying the result obtained in this contribution.

2. New third-order methods

A list of the known second-order methods ($p = 2$) is given:

$$(2.1)\ \phi_1(x) = x - \frac{f(x)}{f'(x)} \quad [1],$$

$$(2.2)\ \phi_2(x) = x - \frac{[f'(x) + 2\lambda f(x)]f(x)}{[f'(x) + \lambda f(x)]^2} \quad \text{for } \lambda \in R, \lambda \neq 0 \quad [2],$$

$$(2.3)\ \phi_3(x) = x - \frac{f(x)f'(x)}{f^2(x) + f'^2(x)} \quad [3].$$

The application of the accelerating procedure Theorem 1.1 with $p = q = 2$ to the second-order methods (2.1), (2.2) and (2.3) produces the new third-order methods, for example,

EXAMPLE 2.1. *Let $\phi(x) = \phi_1(x)$, $\psi(x) = \phi_2(x)$. By Theorem 1.1,*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f\left(x_n - \frac{[f'(x_n)+2\lambda f(x_n)]f(x_n)}{[f'(x_n)+\lambda f(x_n)]^2}\right)}{f'(x_n)}, \tag{2.4}$$

where $\lambda \in R$, $\lambda \neq 0$, is of third order.

EXAMPLE 2.2. *Let $\phi(x) = \phi_1(x)$, $\psi(x) = \phi_3(x)$. By Theorem 1.1,*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f\left(x_n - \frac{f(x_n)f'(x_n)}{f^2(x_n)+f'^2(x_n)}\right)}{f'(x_n)} \tag{2.5}$$

is of third order.

EXAMPLE 2.3. *Let $\phi(x) = \phi_3(x)$, $\psi(x) = \phi_1(x)$. By Theorem 1.1,*

$$x_{n+1} = x_n - \frac{f(x_n)f'(x_n)}{f^2(x_n) + f'^2(x_n)} - \frac{f\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}{f'(x_n)} \tag{2.6}$$

is of third order.

3. Numerical examples

The order of convergence ρ can be approximated using the formula

$$\rho \approx \frac{\ln|(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln|(x_n - \alpha)/(x_{n-1} - \alpha)|}.$$

All computations were done using MAPLE using 64 digit floating point arithmetics (Digits:=64). We accept an approximate solution rather than the exact root, depending on the precision (ϵ) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \epsilon$, ($ii$) $|f(x_{n+1})| < \epsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. For numerical illustrations in this section we used the fixed stopping criterion $\epsilon = 10^{-15}$.

The computational results for various cubically convergent iterative schemes are reported in Table 2. Compared were Newton's method (NM), the method of Weerakoon and Fernando [4] (WF) defined by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - f(x_n)/f'(x_n))}, \tag{3.1}$$

the method derived from midpoint rule [5] (MP) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/(2f'(x_n)))}, \tag{3.2}$$

the method of Homeier [6] (HM) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/f'(x_n))}\right), \tag{3.3}$$

the method of Kou *et al.* [7] (KM) defined by

$$x_{n+1} = x_n - \frac{f(x_n + f(x_n)/f'(x_n)) - f(x_n)}{f'(x_n)}, \tag{3.4}$$

and the methods (2.4) with $\lambda = 1$ (CM1), (2.5) (CM2) and (2.6) (CM3), respectively, introduced in the present contribution. We remark that chosen for comparison are only the methods which do not require the computation of second or higher derivatives of the function to carry out iterations. We used the following test functions:

$$\begin{aligned}
f_1(x) &= x^3 + 4x^2 - 10,\\
f_2(x) &= \sin^2 x - x^2 + 1,\\
f_3(x) &= x^2 - \mathrm{e}^x - 3x + 2,\\
f_4(x) &= \cos x - x,\\
f_5(x) &= (x-1)^3 - 1,\\
f_6(x) &= \sin x - x/2,\\
f_7(x) &= x\mathrm{e}^{x^2} - \sin^2 x + 3\cos x + 5.
\end{aligned}$$

TABLE 1. Comparison of various cubically convergent iterative methods and Newton's method

	IT	COC	x_*	$f(x_*)$	δ
$f_1, x_0 = 1.27$					
NM	5	2	1.3652300134140968457608068290	2.70e-41	1.83e-21
WF	4	3	1.3652300134140968457608068290	0	3.0e-35
MP	4	3	1.3652300134140968457608068290	0	2.60e-36
HM	3	2.96	1.3652300134140968457608068290	-4.45e-48	2.07e-16
KM	4	3	1.3652300134140968457608068290	0	1.77e-33
CM1	4	3	1.3652300134140968457608068290	0	2.04e-29
CM2	4	3	1.3652300134140968457608068290	0	7.28e-31
CM3	4	3	1.3652300134140968457608068290	0	1.70e-25
$f_2, x_0 = 2.0$					
NM	6	2	1.4044916482153412260350868178	-2.26e-32	1.08e-16
WF	5	3	1.4044916482153412260350868178	-2.0e-63	6.02e-42
MP	5	3	1.4044916482153412260350868178	-2.0e-63	7.11e-41
HM	4	3	1.4044916482153412260350868178	-2.0e-63	1.08e-24
KM	5	3	1.4044916482153412260350868178	-2.0e-63	5.29e-31
CM1	4	3.02	1.4044916482153412260350868178	-1.64e-49	6.61e-17
CM2	4	3	1.4044916482153412260350868178	1.30e-63	1.15e-23
CM3	5	2.99	1.4044916482153412260350868178	-1.73e-60	6.78e-21
$f_3, x_0 = 0.5$					
NM	5	2	0.25753028543986076045536730494	3.18e-54	3.0e-27
WF	4	3	0.25753028543986076045536730494	1.0e-63	9.88e-36
MP	4	3	0.25753028543986076045536730494	1.0e-63	9.12e-44
HM	4	3	0.25753028543986076045536730494	-1.0e-63	8.87e-37
KM	4	3	0.25753028543986076045536730494	-1.0e-63	7.43e-29
CM1	4	3	0.25753028543986076045536730494	-2.13e-54	8.30e-19
CM2	4	2.99	0.25753028543986076045536730494	-5.75e-50	2.49e-17
CM3	4	3	0.25753028543986076045536730494	-6.64e-50	2.58e-17
$f_4, x_0 = 0.6$					
NM	5	2	0.73908513321516064165531208767	-2.85e-47	8.78e-24
WF	4	3	0.73908513321516064165531208767	0	2.23e-45
MP	4	3	0.73908513321516064165531208767	0	1.37e-38
HM	4	3	0.73908513321516064165531208767	0	1.03e-42
KM	4	3	0.73908513321516064165531208767	0	1.29e-31
CM1	4	2.99	0.73908513321516064165531208767	0	1.74e-22
CM2	4	3	0.73908513321516064165531208767	1.0e-64	6.65e-24
CM3	4	3	0.73908513321516064165531208767	0	6.35e-23
$f_4, x_0 = 5$					
NM	29	2	0.73908513321516064165531208767	-4.89e-33	1.15e-16
WF	6	3	0.73908513321516064165531208767	0	3.55e-38
MP	82	2.99	0.73908513321516064165531208767	1.0e-64	3.06e-25
HM			divergent		
KM			divergent		
CM1	7	3	0.73908513321516064165531208767	0	1.79e-42
CM2	8	3	0.73908513321516064165531208767	0	5.71e-25
CM3			divergent		

Table 2. Comparison of various cubically convergent iterative methods and Newton's method

	IT	COC	x_*	$f(x_*)$	δ
$f_5, x_0 = 2.4$					
NM	6	2	2	9.87e-33	5.74e-17
WF	5	3	2	0	9.29e-40
MP	5	3	2	0	5.76e-43
HM	4	3	2	3.48e-61	8.87e-21
KM	5	3	2	0	2.17e-38
CM1	5	3	2	0	5.53e-40
CM2	5	3	2	0	3.93e-42
CM3	5	3	2	0	1.03e-26
$f_6, x_0 = 2.3$					
NM	6	2	1.8954942670339809471440357381	-2.45e-48	2.28e-24
WF	4	2.99	1.8954942670339809471440357381	-3.0e-64	1.13e-21
MP	4	2.99	1.8954942670339809471440357381	-1.39e-59	3.64e-20
HM	4	3	1.8954942670339809471440357381	-3.0e-64	2.22e-38
KM	4	2.99	1.8954942670339809471440357381	-3.7e-46	8.27e-16
CM1	4	3	1.8954942670339809471440357381	-3.0e-64	5.88e-28
CM2	4	3	1.8954942670339809471440357381	-2.69e-58	9.98e-20
CM3	5	3	1.8954942670339809471440357381	-3.0e-64	1.03e-32
$f_6, x_0 = 13$					
NM			divergent		
WF	6	3	1.8954942670339809471440357381	1.63e-60	1.87e-20
MP	5	3	1.8954942670339809471440357381	-3.0e-64	2.93e-28
HM			divergent		
KM			divergent		
CM1	72	3	1.8954942670339809471440357381	0	2.39e-37
CM2			divergent		
CM3			divergent		
$f_7, x_0 = -1.1$					
NM	6	2	-1.2076478271309189270094167584	-7.34e-50	4.91e-26
WF	4	3	-1.2076478271309189270094167584	3.33e-56	7.95e-20
MP	4	3	-1.2076478271309189270094167584	-4.0e-63	9.85e-24
HM	4	3	-1.2076478271309189270094167584	-4.0e-63	1.81e-27
KM	4	3	-1.2076478271309189270094167584	-4.0e-63	3.12e-33
CM1	4	3	-1.2076478271309189270094167584	9.35e-55	2.36e-19
CM2	4	3	-1.2076478271309189270094167584	3.53e-56	7.91e-20
CM3	4	3	-1.2076478271309189270094167584	7.35e-45	4.03e-16

As convergence criterion, it was required that the distance of two consecutive approximations δ for the zero was less than 10^{-15}. Also displayed are the number of iterations to approximate the zero (IT), the computational order of convergence (COC), the approximate zero x_*, and the value $f(x_*)$. Note that the approximate zeroes were displayed only up to the 28th decimal places, so it making all looking the same though they may in fact differ.

The test results in Tables 1 and 2 show that the computed order of convergence of the presented iterative methods is three, which agree with the theoretical result developed in this paper. It is well known that convergence of iteration formula is guaranteed only when the initial approximation is sufficiently near the desired root. In general, it may be divergent when the initial approximation is far from the root as this can be observed in Tables 1 and 2. It can be observed that for most of the functions we tested, the methods introduced in the present presentation have at least equal performance compared to the other known methods of the same order.

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Department of Mathematics,
Daejin University,
Pocheon 487-711, Republic of Korea
E-mail: wbkim@daejin.ac.kr

**
Department of Mathematics,
Sungkyunkwan University,
Suwon 440-746, Republic of Korea
E-mail: cbchun@skku.edu

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School of Liberal Arts,
Korea University of Technology and Education,
Cheonan 330-708, Republic of Korea
E-mail: yikim28@kut.ac.kr